

Conserved higher-spin charges in AdS_4

O.A. Gelfond¹ and M.A. Vasiliev²

¹Institute of System Research of Russian Academy of Sciences,
Nakhimovsky prospect 36-1, 117218, Moscow, Russia

²I.E.Tamm Department of Theoretical Physics, Lebedev Physical Institute,
Leninsky prospect 53, 119991, Moscow, Russia

Abstract

Gauge invariant conserved conformal currents built from massless fields of all spins in 4d Minkowski space-time and AdS_4 are described in the unfolded dynamics approach. The current cohomology associated with non-zero conserved charges is found. The resulting list of charges is shown to match the space of parameters of the conformal higher-spin symmetry algebra in four dimensions.

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1 Introduction

Gauge invariant conserved currents of different spins in $4d$ Minkowski space were constructed in [1] in terms of generalized higher-spin (HS) Weyl curvatures introduced originally in [2]. The latter describe on-shell nontrivial gauge invariant combinations of derivatives of fields which generalize the spin-one Maxwell tensor and linearized spin-two Weyl tensor. Later conserved HS currents were also considered in [3, 4] while nontrivial currents leading to non-zero charges were identified in [5].

In [6] it was shown that global conformal HS symmetries of $4d$ massless fields of all spins are described by the Weyl algebra A_4 of eight oscillators. Algebras of symmetries of equations of motion of irreducible free fields and supermultiplets were also found in [6] while extensions to higher dimensions were elaborated in [7, 8, 9, 10].

Closed forms describing the gauge invariant conservation laws in $4d$ Minkowski space-time were found within the unfolded approach in [4]. In this paper we extend these results to AdS_4 and analyze the current cohomology characterizing nontrivial conserved charges. Namely, in [11] it was shown that the space of closed three-forms which can give rise to conserved charges is far larger than the space of HS conserved charges that can be associated with symmetries of massless fields. Hence, it was conjectured in [11] that most of such closed three-forms are exact. In this paper, we show that this is indeed true and that the current cohomology matches the anticipated HS global symmetries. We focus on the gauge invariant currents built in terms of generalized Weyl tensors. Note that non-gauge invariant conserved currents built in terms of HS connections are also available [12] in $4d$ Minkowski space and AdS_4 [13] where it was also shown that the conserved charges associated with these currents are gauge invariant.

A convenient way to analyze conserved quantities is provided by the so-called unfolded approach [14] in which all fields and field equations are formulated in terms of differential forms (for a review see e.g. [15]). The full set of nontrivial symmetry parameters of bilinear conserved currents, that lead to non-zero charges, was found in [16] using the correspondence between unfolded and BRST formulations of $Sp(2M)$ invariant HS equations in the generalized matrix

space-time \mathcal{M}_M . For $M = 2$ the respective charges were shown in [17] to generate the $3d$ $N = 2$ superconformal HS algebra equivalent to the $N = 2$ AdS_4 HS algebra.

An important advantage of the unfolded formulation is that in lower dimensions $d = 3, 4$ dynamical content of a theory is fully characterized by auxiliary spinor variables Y . Transition from fields $C(Y|x)$ to currents $J(Y_1, Y_2|x)$ corresponds to the tensor product of the modules associated with the fields $C(Y|x)$. Technically, all properties of a system are encoded by the Y -dependence that determines in particular the space-time evolution of the system in question. As a result, properties of modules in the Y -space characterize properties of the related physical quantities in the x -space. In the context of the present paper this gives us an opportunity to distinguish between nontrivial and trivial conserved currents in terms of certain algebras acting in the Y -space. Specifically, we identify a Lie algebra $\mathfrak{o}(4, 4)$ that maps any closed three-form associated with a bilinear conserved current in AdS_4 to an exact form, *i.e.*, improvement. As a result, factorization of the generators of this $\mathfrak{o}(4, 4)$ allows one to factor out exact conserved currents. The resulting non-exact bilinear conserved currents in AdS_4 are given in (4.35). Analogous results for Minkowski space are obtained in Section 5 (see (5.7)).

Extension of the construction of gauge invariant conserved charges to AdS_4 performed in this paper may have various applications including in particular computation of HS charges of black hole-like solutions in HS theory [18, 19]. The latter are stationary spherically symmetric solutions of full nonlinear HS field equations that have a singularity in the bulk and fall down properly at space infinity of AdS_4 space-time. Such solutions are analogous to usual GR black holes although their fundamental properties still remain largely unexplored (it is not even clear what kind of horizon (if any) is associated with them). The conserved HS currents introduced in this paper can be used for the perturbative analysis of HS black hole charges. Being associated with closed three-forms, the related charges result from integration over the three-dimensional bulk space. This is different from the construction of two-form currents giving rise to asymptotic charges in AdS as proposed e.g. in [20, 21]. (Two-form charges in HS theory were discussed recently in [22, 23]). On the other hand the both constructions are related because in the area of their applicability the two types of charges coincide pretty much the same way as in the Gauss law in electrodynamics. More generally, the fields contributing to the asymptotic charges are sourced by the currents constructed in this paper, although, being simple at the linearized level, the relation between the two types of charges is far less trivial in the full nonlinear theory.

The rest of the paper is organized as follows. In Section 2 the description of higher-rank fields in the unfolded approach is briefly recalled. In Section 3 we recall the unfolded form of free HS equations in AdS_4 proposed in [14, 24] and their flat limit. In Section 4 conserved HS currents in AdS_4 are constructed in terms of covariantly-constant oscillators and De Rham cohomology of gauge-invariant conserved conformal currents built from massless fields of all spins is found. The resulting nontrivial charges are shown to match the space of parameters of the HS symmetry algebra. In Section 5 conserved currents of $4d$ Minkowski space are reconstructed via the flat limit of those in AdS_4 .

2 Higher-rank fields

Conformal massless fields of all spins in four dimensions can be described [6] by a rank-one zero-form $C(Y|x)$ where x^n are $4d$ space-time coordinates and Y^A are auxiliary spinor variables ($A, B = 1, \dots, 4$ are Majorana spinor indices). It is convenient to interpret $C(Y|x)$ as a vector $|C(Y|x)\rangle$ in the Fock space F of the algebra of oscillators Y^A and Z_A that satisfy commutation relations

$$[Y^A, Y^B] = 0, \quad [Z_A, Z_B] = 0, \quad [Z_A, Y^B] = \delta_A^B. \quad (2.1)$$

The Fock vacua $|0\rangle$ and $\langle 0|$ are defined to obey

$$Z_A|0\rangle = 0, \quad \langle 0|Y^A = 0. \quad (2.2)$$

In these terms the rank-one equation of [6] takes the form

$$D|C(Y|x)\rangle := (d + W(Y, Z|x))|C(Y|x)\rangle = 0, \quad (2.3)$$

where $d = dx^n \frac{\partial}{\partial x^n}$ is De Rham derivative and $W(Y, Z|x)$ satisfies the flatness condition

$$D^2 = 0 : \quad dW + \frac{1}{2}[W, \wedge W] = 0. \quad (2.4)$$

It is convenient to choose $W(Y, Z|x)$ valued in the $\mathfrak{sp}(8)$ realized by bilinears of Y^A and Z_A

$$W(Y, Z|x) = f_{AB}(x) Y^A Y^B + h^{AB}(x) Z_A Z_B + \frac{1}{2} \omega_A^B(x) \{Y^A, Z_B\}, \quad (2.5)$$

where h^{AB} , f_{AB} and ω_A^B are components of the one-form connection. The $\mathfrak{sp}(8)$ flatness conditions are

$$\begin{aligned} R^{AB} &:= d h^{AB} - \omega_C^A \wedge h^{CB} = 0, & R_{AB} &:= d f_{AB} + \omega_A^C \wedge f_{CB} = 0, \\ R_A^B &:= d \omega_A^B + \omega_A^C \wedge \omega_C^B - f_{AC} \wedge h^{CB} = 0. \end{aligned} \quad (2.6)$$

From the oscillator realization it is obvious that massless field equations (2.3) are invariant under the global symmetry associated with the full Weyl algebra of the oscillators Y and Z . Indeed, suppose that $|C(Y|x)\rangle$ solves rank-one equations (2.3). Then any $|\tilde{C}(Y|x)\rangle$ of the form

$$|\tilde{C}(Y|x)\rangle = \eta(Y, Z|x)|C(Y|x)\rangle \quad (2.7)$$

with $\eta(Y, Z|x)$ satisfying

$$D\eta(Y, Z|x) := d\eta(Y, Z|x) + [W(Y, Z|x), \eta(Y, Z|x)] = 0 \quad (2.8)$$

also solves (2.3). Since equations (2.8) are consistent with $D^2 = 0$ by virtue of flatness condition (2.4), their general solution is reconstructed uniquely in terms of $\eta(Y, Z|x_0)$ at any given point x_0 (denoted by 0 in the sequel), hence being characterized by an arbitrary function of Y and Z .

Because $W(Y, Z|x)$ is bilinear in the oscillators Y and Z , equation (2.8) is homogeneous in the oscillators. In particular, one can solve Eq. (2.8) for $\eta(Y, Z|x)$ linear in Y and Z . Clearly, there are eight independent solutions of this type which we denote

$$\mathcal{A}^{\underline{A}}(Y, Z|x), \quad \mathcal{B}_{\underline{B}}(Y, Z|x), \quad (2.9)$$

where $\underline{A}, \underline{B} = 1, \dots, 4$ label independent solutions normalized so that

$$\mathcal{A}^{\underline{A}}(Y, Z|0) = Y^{\underline{A}}, \quad \mathcal{B}_{\underline{A}}(Y, Z|0) = Z_{\underline{A}}. \quad (2.10)$$

This normalization guarantees that $\mathcal{A}^{\underline{A}}(x)$ and $\mathcal{B}_{\underline{B}}(x)$ obey canonical commutation relations at any x

$$[\mathcal{A}^{\underline{A}}, \mathcal{A}^{\underline{B}}] = 0, \quad [\mathcal{B}_{\underline{A}}, \mathcal{B}_{\underline{B}}] = 0, \quad [\mathcal{B}_{\underline{A}}, \mathcal{A}^{\underline{B}}] = \delta_{\underline{A}}^{\underline{B}}. \quad (2.11)$$

Indeed, since the oscillators $\mathcal{A}^{\underline{A}}$ and $\mathcal{B}_{\underline{B}}$ are covariantly constant with respect to D , their commutator is also covariantly constant. Since the commutator of linear combinations of the oscillators $Y^{\underline{A}}$ and $Z_{\underline{B}}$ is independent of the oscillators, D acts on the commutator as d , and hence the right-hand-sides of (2.11) should be coordinate independent.

In terms of $\mathcal{A}^{\underline{A}}$ and $\mathcal{B}_{\underline{A}}$, general solution of equation (2.8) is

$$\eta(Y, Z|x) = \eta(\mathcal{A}, \mathcal{B}), \quad \eta(\mathcal{A}, \mathcal{B}) \equiv \eta(\mathcal{A}(Y, Z|x), \mathcal{B}(Y, Z|x)). \quad (2.12)$$

Thus, in agreement with [6], global conformal HS symmetries form the Lie algebra associated with the Weyl algebra A_4 of four pairs of oscillators. It contains the $\mathfrak{sp}(8)$ subalgebra of bilinears of oscillators. In fact, $\mathcal{A}^{\underline{A}}$ and $\mathcal{B}_{\underline{B}}$ can be interpreted as supergenerators

$$Q^{\underline{A}} = \mathcal{A}^{\underline{A}}, \quad Q_{\underline{B}} = \mathcal{B}_{\underline{B}}, \quad (2.13)$$

which, together with

$$T_{\underline{A}\underline{B}} = Q_{\underline{A}}Q_{\underline{B}}, \quad T^{\underline{A}\underline{B}} = Q^{\underline{A}}Q^{\underline{B}}, \quad T_{\underline{A}}^{\underline{B}} = \frac{1}{2}\{Q_{\underline{A}}, Q^{\underline{B}}\}, \quad (2.14)$$

form $\mathfrak{osp}(1, 8)$.

The rank- r equations for r species of oscillators $Y^{\underline{A}}, Z_{\underline{A}} \rightarrow Y_i^{\underline{A}}, Z_A^j$ ($i, j = 1 \dots r$) can be considered analogously with the Fock space realization of the rank- r field $|C(Y|x)\rangle$

$$D|C(Y|x)\rangle := (d + W^{(r)}(Y, Z|x))|C(Y|x)\rangle = 0, \quad W^{(r)}(Y, Z|x) = \sum_{i=1}^r W(Y_i, Z^i|x). \quad (2.15)$$

Obviously, these equations are invariant under the global symmetry with parameters $\eta(\mathcal{A}, \mathcal{B})$ valued in the Weyl algebra A_{4r} generated by

$$\mathcal{A}_{\underline{A}}^i = \mathcal{A}_{\underline{A}}(Y_i, Z^i|x), \quad \mathcal{B}_i^{\underline{A}} = \mathcal{B}^{\underline{A}}(Y_i, Z^i|x). \quad (2.16)$$

Let $\mathfrak{hs}(n; \mathbb{C})$ be the complex Lie superalgebra resulting from A_n via the \mathbb{Z}_2 graded commutator $[f, g]$ where homogeneous elements $f(Z, Y)$ are associated with even and odd elements,

$f(Z, Y) = (-1)^{\pi_f} f(-Z, -Y)$. Then the (complexified) rank- r equations are invariant under $\mathfrak{hs}(4r; \mathbb{C})$. The rank-one HS algebra $\mathfrak{hs}(4; \mathbb{C})$ belongs to $\mathfrak{hs}(4r; \mathbb{C})$. Among a number of inequivalent embeddings of $\mathfrak{hs}(4; \mathbb{C})$ into $\mathfrak{hs}(4r; \mathbb{C})$, the principal embedding where the element $f(Y, Z) \in \mathfrak{hs}(4; \mathbb{C})$ is represented by $\sum_{i=1}^r f(Y_i, Z_i)$ is most important.

In the 4d Minkowski setup it is convenient to use two-component spinor notation. In these terms,

$$x^{\alpha\alpha'} = x^n \sigma_n^{\alpha\beta'} , \quad Y^A = (y^\alpha, \bar{y}^{\beta'}) , \quad Z_A = (z_\alpha, \bar{z}_{\beta'}) , \quad (2.17)$$

where $\sigma_n^{\alpha\beta'}$ are Hermitian 2×2 matrices and $y^\alpha, z_\alpha, \bar{y}^{\beta'}, \bar{z}_{\beta'}$ are conjugated two-component spinor oscillators with nonzero commutation relations

$$[z_\beta, y^\alpha] = \delta_\beta^\alpha , \quad [\bar{z}_{\alpha'}, \bar{y}^{\beta'}] = \delta_{\alpha'}^{\beta'} . \quad (2.18)$$

Two-component indices are raised and lowered by the symplectic forms $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\alpha'\beta'}$

$$A_\beta = A^\alpha \varepsilon_{\alpha\beta} , \quad A_{\beta'} = A^{\alpha'} \varepsilon_{\alpha'\beta'} , \quad A^\alpha = A_\beta \varepsilon^{\alpha\beta} , \quad A^{\alpha'} = A_{\beta'} \varepsilon^{\alpha'\beta'} . \quad (2.19)$$

The algebra $\mathfrak{u}(2, 2)$ has connections $h^{\alpha\alpha'}, \omega_\alpha^\beta, \bar{\omega}_{\alpha'}^{\beta'}$ and $f_{\alpha\alpha'}$. The Lorentz connection is represented by the traceless parts $\omega_\alpha^{L\beta}$ and $\bar{\omega}_{\alpha'}^{L\beta'}$ of ω_α^β and $\bar{\omega}_{\alpha'}^{\beta'}$, respectively, while the traces are associated with the gauge fields of dilatation b and helicity generator \tilde{b}

$$b = \frac{1}{2} (\omega_\alpha^\alpha + \bar{\omega}_{\alpha'}^{\alpha'}) , \quad \tilde{b} = \frac{1}{2} (\omega_\alpha^\alpha - \bar{\omega}_{\alpha'}^{\alpha'}) . \quad (2.20)$$

The $\mathfrak{u}(2, 2)$ flatness conditions are

$$R^{\alpha\beta'} : = dh^{\alpha\beta'} - \omega_\gamma^\alpha \wedge h^{\gamma\beta'} - \bar{\omega}_{\gamma'}^{\beta'} \wedge h^{\alpha\gamma'} = 0 , \quad (2.21)$$

$$R_{\alpha\beta'} : = df_{\alpha\beta'} + \omega_\alpha^\gamma \wedge f_{\gamma\beta'} + \bar{\omega}_{\beta'}^{\gamma'} \wedge f_{\alpha\gamma'} = 0 , \quad (2.22)$$

$$R_\alpha^\beta : = d\omega_\alpha^\beta + \omega_\alpha^\gamma \wedge \omega_\gamma^\beta - f_{\alpha\gamma'} \wedge h^{\gamma'\beta} = 0 , \quad (2.23)$$

$$\bar{R}_{\alpha'}^{\beta'} : = d\bar{\omega}_{\alpha'}^{\beta'} + \bar{\omega}_{\alpha'}^{\gamma'} \wedge \bar{\omega}_{\gamma'}^{\beta'} - f_{\gamma\alpha'} \wedge h^{\gamma\beta'} = 0 . \quad (2.24)$$

Reduction of unfolded equations (2.3) for the massless field $C(y, \bar{y}|x)$ to the $\mathfrak{u}(2, 2) \subset \mathfrak{sp}(8)$ invariant setup gives

$$D_u^{tw} |C(y, \bar{y}|x)\rangle = 0 , \quad (2.25)$$

where

$$D_u^{tw} = D_{su}^{tw} + \frac{1}{2} \tilde{b} (y^\alpha z_\alpha - \bar{y}^{\alpha'} \bar{z}_{\alpha'}) \quad (2.26)$$

and the conformal covariant derivative D_{su}^{tw} is

$$D_{su}^{tw} = d + \omega_\alpha^{L\beta} y^\alpha z_\beta + \bar{\omega}_{\alpha'}^{L\beta'} \bar{y}^{\alpha'} \bar{z}_{\beta'} + f_{\alpha\alpha'} y^\alpha \bar{y}^{\alpha'} + h^{\alpha\alpha'} z_\alpha \bar{z}_{\beta'} + \frac{1}{2} b (2 + y^\alpha z_\alpha + \bar{y}^{\alpha'} \bar{z}_{\alpha'}) . \quad (2.27)$$

The AdS_4 description with the background fields valued in $\mathfrak{sp}(4) \subset \mathfrak{su}(2, 2)$ results from the Ansatz

$$h^{\alpha\alpha'} = \lambda e^{\alpha\alpha'} , \quad f_{\alpha\alpha'} = \lambda e_{\alpha\alpha'} , \quad b = \tilde{b} = 0 \quad (2.28)$$

and

$$D_{ads}^{tw} = D^L + \lambda e^{\alpha\alpha'} y_\alpha \bar{y}_{\alpha'} + \lambda e^{\alpha\alpha'} z_\alpha \bar{z}_{\alpha'} , \quad D^L = d + \left(\omega^{L\alpha\beta} y_\alpha z_\beta + \bar{\omega}^{L\alpha'\beta'} \bar{y}_{\alpha'} \bar{z}_{\beta'} \right). \quad (2.29)$$

The rank-one unfolded equation in AdS_4 is

$$D_{ads}^{tw} |C(y, \bar{y}|x)\rangle = 0. \quad (2.30)$$

3 Rank-two equations and covariant oscillators in AdS_4

In this paper we focus on the rank-two field $|J(Y|x)\rangle$ associated with $4d$ conformal conserved currents built from massless fields of all spins in [11]. It satisfies the rank-two *current* equation

$$D_2 |J(Y|x)\rangle := (d + \sum_{i=1,2} W(Y_i, Z^i|x)) |J(Y|x)\rangle = 0 \quad (3.1)$$

with $W(Y, Z|x)$ (2.5).

Let us look for a three-form ω closed by virtue of rank-two equations in the form

$$\omega = \langle \Omega | J \rangle, \quad (3.2)$$

where $\langle \Omega |$ is a three-form that verifies the equation

$$d\langle \Omega | + \langle \Omega | \sum_{i=1,2} W(Y_i, Z^i|x) = 0, \quad (3.3)$$

which, together with (3.1), implies

$$d\omega = 0. \quad (3.4)$$

On-shell closed forms generate conserved charges

$$Q(\omega) = \int_{\Sigma^3} \omega. \quad (3.5)$$

Conservation means that Q is independent of local variations of Σ^3 such as variation of time. Exact ω do not contribute to Q for solutions of the field equations that decrease fast enough at space infinity. Hence, nontrivial charges Q are associated with the *current cohomology*.

Clearly, a three-form

$$\omega(\eta(\mathcal{A}, \mathcal{B})) = \langle \Omega | \eta(\mathcal{A}, \mathcal{B}) | J \rangle, \quad (3.6)$$

where $\eta(\mathcal{A}, \mathcal{B})$ is any function of the oscillators $\mathcal{A}_{\underline{A}}^i$ and $\mathcal{B}_{\underline{A}}^i$ (2.16) with $i = 1, 2$, is also closed. Hence, the full space of closed *current forms* is the space of arbitrary functions of oscillators (2.16). This freedom should encode the freedom in different HS charges. Indeed, as shown in [4], the realization of a rank-two field in terms of bilinears of rank-one fields gives rise to the full list of conformal gauge invariant $4d$ conserved currents of all spins identified as generalized Bell-Robinson currents in [1]. However, the freedom in a function of two sets of oscillators $\mathcal{A}_{\underline{A}}^i$

and \mathcal{B}_i^A is far larger than that in HS symmetries of rank-one equations, parametrized by a function of the rank-one variables \mathcal{A}_A and \mathcal{B}^A (2.9). Hence, we conjectured in [11] that most of the closed forms (3.6) are exact, generating no nontrivial HS charges.

Using the correspondence between unfolded and BRST formulations of $Sp(2M)$ invariant HS field equations in the generalized matrix space-time \mathcal{M}_M with any M the space of nontrivial parameters η and $\tilde{\eta}$ of bilinear conserved currents that lead to nonzero charges Q_η and $\tilde{Q}_{\tilde{\eta}}$ was found in [16]. The latter were shown in [17] to generate the supersymmetric HS algebra with the doubled number of charges of a given spin.¹ The identification of nontrivial conserved charges in the flat Minkowski space was also done in [5] by a different method.

Let us briefly recall the construction of [16]. Consider a generalized space-time $\mathcal{M}_M(X)$ with matrix coordinates $X^{AB} = X^{BA}$, $A, B = 1, \dots, M$. M -forms in $\mathcal{M}_M(X) \times \mathbb{R}^{2M}(U, V)$ [25]

$$\Phi_\eta(J) = \frac{1}{2} \left(i\hbar d X^{AB} \frac{\partial}{\partial U^B} + d V^A \right)^M \eta(\mathfrak{B}, \tilde{\mathfrak{B}}) J(U, V | X) \Big|_{U=0} \quad (3.7)$$

and

$$\tilde{\Phi}_{\tilde{\eta}}(J) = \frac{1}{2} \left(i\hbar d X^{AB} \frac{\partial}{\partial V^B} + d U^A \right)^M \tilde{\eta}(\mathfrak{B}, \tilde{\mathfrak{B}}) J(U, V | X) \Big|_{V=0} \quad (3.8)$$

are closed by virtue of the current equation [26]

$$\left(\frac{\partial}{\partial X^{AB}} - i\hbar \frac{\partial^2}{\partial U^{(A} \partial V^{B)}} \right) J(U, V | X) = 0 \quad (3.9)$$

with parameters η and $\tilde{\eta}$ polynomial in the operators \mathfrak{B} and $\tilde{\mathfrak{B}}$, respectively,

$$\mathfrak{B}_A(U, V | X) = \frac{\partial}{\partial U^A}, \quad \mathfrak{B}^B(U, V | X) = i\hbar X^{AB} \frac{\partial}{\partial U^A} + V^B, \quad (3.10)$$

$$\tilde{\mathfrak{B}}_A(U, V | X) = \frac{\partial}{\partial V^A}, \quad \tilde{\mathfrak{B}}^B(U, V | X) = i\hbar X^{AB} \frac{\partial}{\partial V^A} + U^B, \quad (3.11)$$

which satisfy

$$\left[\frac{\partial}{\partial X^{AB}} - i\hbar \frac{\partial^2}{\partial U^{(A} \partial V^{B)}} , \mathfrak{B} \right] = \left[\frac{\partial}{\partial X^{AB}} - i\hbar \frac{\partial^2}{\partial U^{(A} \partial V^{B)}} , \tilde{\mathfrak{B}} \right] = 0. \quad (3.12)$$

In [16], it was shown that non-exact forms Φ (3.7) and $\tilde{\Phi}$ (3.8) are represented, respectively, by $\tilde{\mathfrak{B}}$ -independent $\eta(\mathfrak{B}(U, V | X))$ and \mathfrak{B} -independent $\tilde{\eta}(\tilde{\mathfrak{B}}(U, V | X))$ which can be interpreted as parameters of global HS symmetry transformations generated by the charges

$$Q(J_{\eta(\mathfrak{B})}) = \int \frac{1}{2} \left(i\hbar d X^{AB} \frac{\partial}{\partial U^B} + d V^A \right)^M \eta(\mathfrak{B}) J(U, V | X) \Big|_{U=V=0}, \quad (3.13)$$

$$\tilde{Q}(J_{\tilde{\eta}(\tilde{\mathfrak{B}})}) = \int \frac{1}{2} \left(i\hbar d X^{AB} \frac{\partial}{\partial V^B} + d U^A \right)^M \tilde{\eta}(\tilde{\mathfrak{B}}) J(U, V | X) \Big|_{U=V=0}, \quad (3.14)$$

¹Note however that in [17], where J were realized as bilinears of physical fields, parameters of currents satisfied appropriate symmetry conditions.

where the integration is over some M -dimensional cycle in $\mathcal{M}_M \times \mathbb{R}^{2M}$. For J bilinear in rank-one fields C_i , where $i = 1, \dots, N$ are color indices, the charges generate the HS algebra [17].

As explained in [25], the proper restriction of (3.13) and (3.14) at $M = 4$ to $4d$ Minkowski space with local coordinates $x^{\alpha\alpha'}$ gives Minkowski conserved charges.

In this paper we construct the full list of gauge invariant conserved currents in AdS_4 . Introducing the two-component spinor oscillators obeying

$$[z_\beta^i, y_j^\alpha] = \delta_\beta^\alpha \delta_j^i, \quad [\bar{z}_{\alpha'}^i, \bar{y}_j^{\beta'}] = \delta_{\alpha'}^{\beta'} \delta_j^i \quad (3.15)$$

it is convenient to rescale $y_2, \bar{y}_2 \rightarrow iy_2, i\bar{y}_2$ and $z_2, \bar{z}_2 \rightarrow -iz_2, -i\bar{z}_2$ to obtain from (3.1)

$$D_{2ads}^{tw} |J(Y|x)\rangle = 0, \quad D_{2ads}^{tw} = D_2^L + \widetilde{W}^{(2)}, \quad (3.16)$$

$$D_2^L = d + \omega^{L\alpha\beta} (y_{1\alpha} z_\beta^1 + y_{2\alpha} z_\beta^2) + \bar{\omega}^{L\alpha'\beta'} (\bar{y}_{1\alpha'} \bar{z}_{\beta'}^1 + \bar{y}_{2\alpha'} \bar{z}_{\beta'}^2), \quad (3.17)$$

$$\widetilde{W}^{(2)} = \lambda e^{\alpha\alpha'} \left(y_{1\alpha} \bar{y}_{1\alpha'} + z_\alpha^1 \bar{z}_{\alpha'}^1 - y_{2\alpha} \bar{y}_{2\alpha'} - z_\alpha^2 \bar{z}_{\alpha'}^2 \right). \quad (3.18)$$

Packing the oscillators $y_i^\alpha, \bar{y}_i^{\alpha'}, z_\alpha^i, \bar{z}_{\alpha'}^i$ into $\kappa_\alpha^{n\hat{n}}, \zeta_{\alpha'}^{m\hat{n}}$ with $n = -, +$ and $\hat{n} = \hat{-}, \hat{+}$ by setting

$$\begin{aligned} \kappa_\alpha^{+\hat{+}} &= y_{2\alpha}, & \kappa_\alpha^{+\hat{-}} &= -y_{1\alpha}, & \kappa_\alpha^{-\hat{+}} &= z_\alpha^1, & \kappa_\alpha^{-\hat{-}} &= z_\alpha^2, \\ \zeta_{\alpha'}^{+\hat{+}} &= -\bar{z}_{\alpha'}^2, & \zeta_{\alpha'}^{+\hat{-}} &= -\bar{z}_{\alpha'}^1, & \zeta_{\alpha'}^{-\hat{+}} &= \bar{y}_{1\alpha'}, & \zeta_{\alpha'}^{-\hat{-}} &= -\bar{y}_{2\alpha'}, \end{aligned} \quad (3.19)$$

one can see that nonzero commutators acquire the form

$$[\kappa_\beta^{n\hat{k}}, \kappa_\alpha^{m\hat{n}}] = \varepsilon^{nm} \varepsilon^{\hat{k}\hat{n}} \varepsilon_{\beta\alpha}, \quad [\zeta_{\beta'}^{n\hat{k}}, \zeta_{\alpha'}^{m\hat{n}}] = \varepsilon^{nm} \varepsilon^{\hat{k}\hat{n}} \varepsilon_{\beta'\alpha'}, \quad (3.20)$$

where indices are raised and lowered by $\varepsilon_{nm}, \varepsilon_{\hat{n}\hat{m}}, \varepsilon^{nm}$ and $\varepsilon^{\hat{n}\hat{m}}$ with

$$\varepsilon_{-+} = -\varepsilon_{+-} = 1, \quad \varepsilon^{-+} = -\varepsilon^{+-} = 1, \quad \varepsilon_{-\hat{+}} = -\varepsilon_{\hat{+}-} = 1, \quad \varepsilon^{\hat{-}\hat{+}} = -\varepsilon^{\hat{+}\hat{-}} = 1. \quad (3.21)$$

From (2.2) it follows

$$\langle 0 | \kappa_\alpha^{+\hat{m}} = 0, \quad \langle 0 | \zeta_{\alpha'}^{-\hat{m}} = 0, \quad \kappa_\alpha^{-\hat{m}} | 0 \rangle = 0, \quad \zeta_{\alpha'}^{+\hat{m}} | 0 \rangle = 0. \quad (3.22)$$

In these terms, Eq. (3.18) takes the form

$$\widetilde{W}^{(2)}(\kappa, \zeta|x) = \lambda e^{\alpha\beta'} \kappa_\alpha^{m\hat{n}} \zeta_{\beta' m \hat{n}}. \quad (3.23)$$

Analogously, the covariantly constant oscillators with respect to the rank-two covariant derivative D_{2ads}^{tw} (3.16) are packed into

$$\tau_a^{n\hat{n}}(\kappa, \zeta|x), \quad v_{a'}^{n\hat{n}}(\kappa, \zeta|x), \quad a = 1, 2; \quad a' = 1, 2 \quad (3.24)$$

$$[D_{2ads}^{tw}, \tau_a^{n\hat{n}}(\kappa, \zeta|x)] = [D_{2ads}^{tw}, v_{a'}^{n\hat{n}}(\kappa, \zeta|x)] = 0$$

so that

$$\tau_a^{m\hat{n}}(\kappa, \zeta|0) = \kappa_\alpha^{m\hat{n}} \delta_a^\alpha, \quad v_{b'}^{m\hat{n}}(\kappa, \zeta|0) = \zeta_{\beta'}^{m\hat{n}} \delta_{b'}^{\beta'}. \quad (3.25)$$

Eq. (3.25) guarantees that $\tau(x)$ and $v(x)$ obey analogous commutation relations at any x

$$[\tau_b^{n\hat{k}}(x), \tau_a^{m\hat{n}}(x)] = \varepsilon^{nm} \varepsilon^{\hat{k}\hat{n}} \varepsilon_{ba}, \quad [v_{b'}^{n\hat{k}}(x), v_{a'}^{m\hat{n}}(x)] = \varepsilon^{nm} \varepsilon^{\hat{k}\hat{n}} \varepsilon_{b'a'}, \quad [\tau_b^{n\hat{k}}(x), v_{a'}^{m\hat{n}}(x)] = 0. \quad (3.26)$$

This can also be seen in terms of Killing spinors $c^\beta(x)$ and $s^{\beta'}(x)$ of [11], that obey

$$D^L c^\alpha(x) + \lambda e^{\alpha\beta'} s_{\beta'}(x) = 0, \quad D^L s^{\beta'}(x) + \lambda e^{\alpha\beta'} c_\alpha(x) = 0. \quad (3.27)$$

A basis of the space of solutions of this system is formed by four independent pairs of spinors $(c_a^\beta(x), s_a^{\beta'}(x))$ and $(c_{a'}^\beta(x), s_{a'}^{\beta'}(x))$ labeled by $a = 1, 2$ and $a' = 1, 2$ and obeying the conditions

$$c_a^\beta(0) = \delta_a^\beta, \quad s_a^{\beta'}(0) = 0, \quad c_{a'}^\beta(0) = 0, \quad s_{a'}^{\beta'}(0) = \delta_{a'}^{\beta'} \quad (3.28)$$

implying that

$$\overline{c_a^\beta(x)} = s_{a'}^{\beta'}(x), \quad \overline{s_a^{\beta'}(x)} = c_{a'}^\beta(x).$$

A specific form of the Killing spinors depends on a chosen coordinate system. Covariantly constant oscillators τ, v (3.24) are expressed via the Killing spinors as

$$\tau_a^{m\hat{k}}(\kappa, \zeta|x) = c_a^\gamma(x) \kappa_\gamma^{m\hat{k}} - s_a^{\gamma'}(x) \zeta_{\gamma'}^{m\hat{k}}, \quad v_{a'}^{m\hat{k}}(\kappa, \zeta|x) = -c_{a'}^\gamma(x) \kappa_\gamma^{m\hat{k}} + s_{a'}^{\gamma'}(x) \zeta_{\gamma'}^{m\hat{k}}. \quad (3.29)$$

Using (3.20) and (3.23) we observe that the operators

$$f^{nm} = \frac{1}{4} \{ \kappa_\beta^n \hat{m}, \kappa^{\beta m} \hat{m} \} + \frac{1}{4} \{ \zeta_{\beta'}^n \hat{m}, \zeta^{\beta' m} \hat{m} \}, \quad (3.30)$$

$$g^{\hat{n}\hat{m}} = \frac{1}{4} \{ \kappa_{\beta k} \hat{n}, \kappa^{\beta k} \hat{m} \} + \frac{1}{4} \{ \zeta_{\beta' k} \hat{n}, \zeta^{\beta' k} \hat{m} \} \quad (3.31)$$

are covariantly constant with respect to the rank-two covariant derivative D_{2ads}^{tw} (3.16) forming two mutually commutative $\mathfrak{su}(2)$ algebras. Algebras (3.30) and (3.31) will be referred to as vertical ${}^v\mathfrak{su}(2)$ and horizontal ${}^h\mathfrak{su}(2)$, respectively. The standard bases f_j of ${}^v\mathfrak{su}(2)$ and g_j of ${}^h\mathfrak{su}(2)$ are

$$f_- = -f^{--} = z^1_\alpha z^{2\alpha} + \bar{y}_1^{\alpha'} \bar{y}^2_{\alpha'}, \quad f_+ = f^{++} = y_1^\alpha y_{2\alpha} + \bar{z}^1_{\alpha'} \bar{z}^{2\alpha'}, \quad f_0 = 2f^{-+} = 2H_2 + 2H_1, \quad (3.32)$$

$$g_- = -g^{\hat{-}\hat{-}} = y_1^\alpha z^2_\alpha + \bar{y}_2^{\alpha'} \bar{z}^1_{\alpha'}, \quad g_+ = g^{\hat{+}\hat{+}} = y_2^\alpha z^1_\alpha + \bar{y}_1^{\alpha'} \bar{z}^2_{\alpha'}, \quad g_0 = 2g^{\hat{-}\hat{+}} = 2H_2 - 2H_1,$$

$$H_1 = \frac{1}{2} (y_1^\alpha z^1_\alpha - \bar{y}_1^{\alpha'} \bar{z}^1_{\alpha'}), \quad H_2 = \frac{1}{2} (y_2^\alpha z^2_\alpha - \bar{y}_2^{\alpha'} \bar{z}^2_{\alpha'}).$$

The ${}^v\mathfrak{su}(2)$ and ${}^h\mathfrak{su}(2)$ act on κ and ζ as follows

$$[f^{mn}, \kappa_\beta^{k\hat{m}}] = \frac{1}{2} \varepsilon^{mk} \kappa_\beta^{n\hat{m}} + \frac{1}{2} \varepsilon^{nk} \kappa_\beta^{m\hat{m}}, \quad [f^{mn}, \zeta_{\beta'}^{k\hat{m}}] = \frac{1}{2} \varepsilon^{mk} \zeta_{\beta'}^{n\hat{m}} + \frac{1}{2} \varepsilon^{nk} \zeta_{\beta'}^{m\hat{m}}, \quad (3.33)$$

$$[g^{\hat{m}\hat{n}}, \kappa_\beta^{m\hat{k}}] = \frac{1}{2} \varepsilon^{\hat{m}\hat{k}} \kappa_\beta^{m\hat{n}} + \frac{1}{2} \varepsilon^{\hat{n}\hat{k}} \kappa_\beta^{m\hat{m}}, \quad [g^{\hat{m}\hat{n}}, \zeta_{\beta'}^{m\hat{k}}] = \frac{1}{2} \varepsilon^{\hat{m}\hat{k}} \zeta_{\beta'}^{m\hat{n}} + \frac{1}{2} \varepsilon^{\hat{n}\hat{k}} \zeta_{\beta'}^{m\hat{m}}.$$

From here it follows that ${}^h\mathfrak{su}(2)$ and ${}^v\mathfrak{su}(2)$ act on the oscillators τ and v (3.29) analogously

$$\begin{aligned} [f^{mn}, \tau_\beta^{k\hat{m}}] &= \frac{1}{2}\varepsilon^{mk}\tau_\beta^{n\hat{m}} + \frac{1}{2}\varepsilon^{nk}\tau_\beta^{m\hat{m}}, & [f^{mn}, v_{\beta'}^{k\hat{m}}] &= \frac{1}{2}\varepsilon^{mk}v_{\beta'}^{n\hat{m}} + \frac{1}{2}\varepsilon^{nk}v_{\beta'}^{m\hat{m}}, \\ [g^{\hat{m}\hat{n}}, \tau_\beta^{m\hat{k}}] &= \frac{1}{2}\varepsilon^{\hat{m}\hat{k}}\tau_\beta^{n\hat{m}} + \frac{1}{2}\varepsilon^{\hat{n}\hat{k}}\tau_\beta^{m\hat{m}}, & [g^{\hat{m}\hat{n}}, v_{\beta'}^{m\hat{k}}] &= \frac{1}{2}\varepsilon^{\hat{m}\hat{k}}v_{\beta'}^{n\hat{m}} + \frac{1}{2}\varepsilon^{\hat{n}\hat{k}}v_{\beta'}^{m\hat{m}}. \end{aligned} \quad (3.34)$$

Indeed, being covariantly constant, $f \in {}^v\mathfrak{su}(2)$ and $g \in {}^h\mathfrak{su}(2)$ keep the same form in terms of $\tau_a^{m\hat{m}}$ and $v_{a'}^{m\hat{k}}$,

$$f^{nm} = \frac{1}{4}\{\tau_b^{n\hat{m}}, \tau^{bm\hat{m}}\} + \frac{1}{4}\{v_{a'}^{n\hat{m}}, v^{a'm\hat{m}}\}, \quad (3.35)$$

$$g^{\hat{n}\hat{m}} = \frac{1}{4}\{\tau_{bk}^{\hat{n}}, \tau^{bk\hat{m}}\} + \frac{1}{4}\{v_{a'k}^{\hat{n}}, v^{a'k\hat{m}}\}. \quad (3.36)$$

4 On-shell de Rham cohomology of AdS_4 currents

Let us show that the following three-forms $\langle \Omega^{\hat{m}\hat{k}} | = \langle \Omega^{\hat{k}\hat{m}} | = \langle 0 | \Omega^{\hat{m}\hat{k}}$ with

$$\Omega^{\hat{m}\hat{k}} := \mathcal{H}^{\alpha\alpha'} \{ \kappa_\alpha^{-\hat{m}} \zeta_{\alpha' - \hat{k}} + \kappa_\alpha^{-\hat{k}} \zeta_{\alpha' - \hat{m}} \}, \quad \mathcal{H}^{\alpha\delta'} = -\frac{1}{3}e^\alpha_{\alpha'} e^{\beta\alpha'} e_\beta^{\delta'} \quad (4.1)$$

obey (3.3). Indeed, using (3.22) and

$$\mathcal{H}^{\eta\beta'} e^{\gamma\gamma'} = \varepsilon^{\eta\gamma} \varepsilon^{\beta'\gamma'} \mathbf{H}, \quad \mathbf{H} := \frac{1}{4} e_{\eta\sigma'} \mathcal{H}^{\eta\sigma'}, \quad (4.2)$$

Eqs. (3.20) and (3.23) yield

$$\begin{aligned} d\langle \Omega^{\hat{m}\hat{k}} | + \langle \Omega^{\hat{m}\hat{k}} | \widetilde{W}^{(2)}(\kappa, \zeta|x) &= \langle 0 | [\widetilde{W}^{(2)}(\kappa, \zeta|x), \Omega^{\hat{m}\hat{k}}] + \langle \Omega^{\hat{m}\hat{k}} | \widetilde{W}^{(2)}(\kappa, \zeta|x) \\ &= \langle 0 | \widetilde{W}^{(2)}(\kappa, \zeta|x) \Omega^{\hat{m}\hat{k}} = \lambda \mathbf{H} \langle 0 | \kappa_\beta^{-\hat{n}} \zeta_{\beta' - \hat{n}} \{ \kappa^{\beta - \hat{m}} \zeta^{\beta' - \hat{k}} + \kappa^{\beta - \hat{k}} \zeta^{\beta' - \hat{m}} \}. \end{aligned} \quad (4.3)$$

The last term is zero since being symmetric in \hat{k} and \hat{m} it turns out to be antisymmetric by virtue of (3.20). Then the form

$$\omega = \langle \Omega^{\hat{m}\hat{k}} | \eta_{\hat{m}\hat{k}}(\tau, v) | J(Y|x) \rangle \quad (4.4)$$

is closed for any $\eta_{\hat{m}\hat{k}}$ provided that $J(Y|x)$ verifies current equation (3.16).

The central fact of the analysis of the current cohomology is that each of the forms

$$\omega_a^{m\hat{m}} = \langle \Omega^{\hat{m}\hat{k}} | \tau_a^{m\hat{k}} \eta(\tau, v) | J(Y|x) \rangle, \quad \omega_{b'}^{m\hat{m}} = \langle \Omega^{\hat{m}\hat{k}} | v_{b'}^{m\hat{k}} \eta(\tau, v) | J(Y|x) \rangle \quad (4.5)$$

is exact provided that $|J(Y|x)\rangle$ solves (3.16).

For instance, let us prove that $\omega_a^{m\hat{m}}$ (4.5) with $\eta = 1$ is exact (other cases are analogous). Indeed, using that

$$H^{\alpha\beta} e^{\gamma\delta'} = \varepsilon^{\alpha\gamma} \mathcal{H}^{\beta\delta'} + \varepsilon^{\beta\gamma} \mathcal{H}^{\alpha\delta'}, \quad H^{\alpha\beta} = e^\alpha_{\alpha'} e^{\beta\alpha'},$$

we obtain by virtue of Eq. (3.27)

$$\begin{aligned} dE_b^{-\hat{m}} &:= d\langle 0 | H^{\alpha\beta} c_{b\alpha} \kappa_{\beta}^{-\hat{m}} | J(Y|x) \rangle = \langle 0 | H^{\alpha\beta} (dc_{b\alpha}) \kappa_{\beta}^{-\hat{m}} - H^{\alpha\beta} c_{b\alpha} \kappa_{\beta}^{-\hat{m}} \widetilde{W}^{(2)}(\kappa, \zeta|x) | J(Y|x) \rangle \\ &= \lambda \langle 0 | 3\mathcal{H}^{\beta\beta'} s_{b\beta'}(x) \kappa_{\beta}^{-\hat{m}} + (\varepsilon^{\mu\alpha} \mathcal{H}^{\beta\nu'} + \varepsilon^{\mu\beta} \mathcal{H}^{\alpha\nu'}) c_{b\alpha} \kappa_{\beta}^{-\hat{m}} (\kappa_{\mu}^{+\hat{n}} \zeta_{\nu'+\hat{n}} + \kappa_{\mu}^{-\hat{n}} \zeta_{\nu'-\hat{n}}) | J(Y|x) \rangle. \end{aligned} \quad (4.6)$$

Neglecting the term with $\zeta_{\nu'+\hat{n}}$ by virtue of (3.22) and using (3.29) along with

$$[\Omega^{\hat{m}\hat{k}}, \tau_a^{-\hat{k}}] = [\mathcal{H}^{\alpha\alpha'} \{\kappa_{\alpha}^{-\hat{m}} \zeta_{\alpha'-\hat{k}} + \kappa_{\alpha}^{-\hat{k}} \zeta_{\alpha'-\hat{m}}\}, \tau_a^{-\hat{k}}] = -3\mathcal{H}^{\alpha\alpha'} s_{a\alpha'}(x) \kappa_{\alpha}^{-\hat{m}}, \quad (4.7)$$

(4.6) yields

$$\begin{aligned} dE_b^{-\hat{m}} &= \lambda \langle 0 | -[\Omega^{\hat{m}\hat{k}}, \tau_b^{-\hat{k}}] + \mathcal{H}^{\beta\nu'} c_b^{\mu} \kappa_{\beta}^{-\hat{m}} \kappa_{\mu}^{-\hat{n}} \zeta_{\nu'-\hat{n}} + \mathcal{H}^{\beta\nu'} c_b^{\mu} \kappa_{\beta}^{\mu-\hat{m}} \kappa_{\mu}^{-\hat{n}} \zeta_{\nu'-\hat{n}} | J(Y|x) \rangle \\ &= \lambda \langle 0 | -\Omega^{\hat{m}\hat{k}} \tau_b^{-\hat{k}} + \mathcal{H}^{\alpha\alpha'} c_a^{\gamma}(x) \kappa_{\gamma}^{-\hat{k}} \{\kappa_{\alpha}^{-\hat{m}} \zeta_{\alpha'-\hat{k}} + \kappa_{\alpha}^{-\hat{k}} \zeta_{\alpha'-\hat{m}}\} \\ &\quad + \mathcal{H}^{\beta\nu'} c_b^{\mu} \kappa_{\mu}^{-\hat{n}} \{\kappa_{\beta}^{-\hat{m}} \zeta_{\nu'-\hat{n}} + \kappa_{\beta}^{-\hat{n}} \zeta_{\nu'-\hat{m}}\} | J(Y|x) \rangle = -\lambda \langle 0 | \Omega^{\hat{m}\hat{k}} \tau_b^{-\hat{k}} | J(Y|x) \rangle, \end{aligned} \quad (4.8)$$

where we used that $\kappa_{\alpha}^{-\hat{n}} \kappa_{\beta}^{-\hat{n}} = \frac{1}{2} \varepsilon_{\alpha\beta} \kappa_{\gamma}^{-\hat{n}} \kappa_{\gamma}^{-\hat{n}}$.

Analogously one can show that other forms (4.5) are exact. An important consequence of this fact and commutation relations (3.26) is that the following forms are also exact:

$$\langle \Omega^{\hat{m}\hat{n}} | \{\tau_a^m \tau_b^{k\hat{k}}\} \eta(\tau, v) | J \rangle, \quad \langle \Omega^{\hat{m}\hat{n}} | \{v_a^m \tau_b^{k\hat{k}}\} \eta(\tau, v) | J \rangle, \quad \langle \Omega^{\hat{m}\hat{n}} | \{\tau_a^m v_b^{k\hat{k}}\} \eta(\tau, v) | J \rangle. \quad (4.9)$$

For instance, since $\langle \Omega^{\hat{m}\hat{n}} | (\tau_a^m \tau_b^{k\hat{k}}) \eta(\tau, v) | J \rangle$ is exact, the form

$$\begin{aligned} \langle \Omega^{\hat{m}\hat{n}} | (\tau_a^m \tau_b^{k\hat{k}} - \tau_b^{k\hat{k}} \tau_a^m) \eta(\tau, v) | J \rangle &= \\ \frac{1}{2} \langle \Omega^{\hat{m}\hat{n}} | (\{\tau_a^m \tau_b^{k\hat{k}}\} - \{\tau_b^{k\hat{k}} \tau_a^m\}) \eta(\tau, v) | J \rangle &= \frac{1}{2} \varepsilon^{\hat{n}\hat{k}} \langle \Omega^{\hat{m}\hat{n}} | \{\tau_a^m \tau_b^{k\hat{k}}\} | J \rangle \end{aligned} \quad (4.10)$$

is exact. The proof for the other forms in (4.9) is analogous.

This fact admits the following interpretation. Various bilinears in τ_a^m and $v_b^{k\hat{k}}$ form a Lie algebra $\mathfrak{sp}(16)$ while

$$\mathcal{G}_{ab}^{mk} = \frac{1}{2} \{\tau_a^m \tau_b^{k\hat{k}}\}, \quad \mathcal{G}_{ab'}^{mk} = \frac{1}{2} \{\tau_a^m v_{b'}^{k\hat{k}}\}, \quad \mathcal{G}_{a'b'}^{mk} = \frac{1}{2} \{v_{a'}^m v_{b'}^{k\hat{k}}\} \quad (4.11)$$

form a Lie algebra $\mathfrak{o}(4, 4)$ that commutes with the horizontal ${}^h\mathfrak{su}(2)$ (3.31) acting on the hatted indices. For parameters η polynomial in oscillators, factorization of generators (4.11) allows us to factor out any combination of oscillators containing antisymmetrization of a pair of the hatted Latin indices.² The remaining forms $\widetilde{\omega}$ (4.4) have totally symmetrized hatted indices.

²Beyond the space of polynomials the situation is different because, formally, it follows that all nontrivial $\mathfrak{o}(4, 4)$ -modules should be factored out. However, this ‘exact’ representation for conserved currents turns out to be space-time nonlocal, containing infinitely many derivatives, giving rise to quasi-exact representations analogous to those considered in [27].

To describe such forms consider the space \mathbf{P}_{AdS} of *preforms*

$$\Omega^{\hat{n}, \hat{m}} \eta(\tau_a^{m\hat{k}}, v_{a'}^{n\hat{p}}) \quad (4.12)$$

with totally symmetrized hatted indices, including those of $\Omega^{\hat{n}, \hat{m}}$. Clearly, ${}^h\mathfrak{su}(2)$ (3.31) leaves \mathbf{P}_{AdS} invariant. Since any ${}^h\mathfrak{su}(2)$ highest vector in \mathbf{P}_{AdS} that is symmetric in hatted indices has the form

$$\Omega^{\hat{+}\hat{+}} \eta(\tau_a^{m\hat{+}}, v_{a'}^{n\hat{+}}), \quad (4.13)$$

\mathbf{P}_{AdS} is a span of vectors

$$\Lambda_N(\eta) = \text{ad}_{\mathfrak{g}_-}^N \left(\Omega^{\hat{+}\hat{+}} \eta(\tau_a^{m\hat{+}}, v_{a'}^{n\hat{+}}) \right), \quad \text{ad}_x(y) = [x, y], \quad \mathfrak{g}_- \in {}^h\mathfrak{su}(2) \quad (4.14)$$

with various $N \geq 0$, *i.e.*,

$$\mathbf{P}_{AdS} = \sum_N \oplus \Lambda_N(\eta). \quad (4.15)$$

Now we observe that the Cartan element $\mathfrak{g}_0 \in {}^h\mathfrak{su}(2)$ (3.32) annihilates both $|0\rangle$ and $\langle 0|$. Hence,

$$\langle 0 | [\mathfrak{g}_0, \Lambda_N(\eta)J] | 0 \rangle = 0 \quad (4.16)$$

for any $\Lambda_N(\eta)$ (4.14) and J . This implies that $\langle 0 | \Lambda_N(\eta)J | 0 \rangle$ can be nonzero only if

$$[\mathfrak{g}_0, \Lambda_N(\eta)J] = 0. \quad (4.17)$$

Evidently any current $J(Y|x)$ and parameter $\eta(\tau, v)$ can be decomposed as

$$J = \sum_{\mu} J_{\mu}, \quad \eta = \sum_{\nu} \eta_{\nu}, \quad [\mathfrak{g}_0, J_{\mu}] = \mu J_{\mu}, \quad [\mathfrak{g}_0, \eta_{\nu}] = \nu \eta_{\nu}, \quad \mu, \nu \in \mathbb{Z}. \quad (4.18)$$

Since

$$[\mathfrak{g}_0, \tau_a^{m\hat{+}}] = \tau_a^{m\hat{+}}, \quad [\mathfrak{g}_0, v_{a'}^{m\hat{+}}] = v_{a'}^{m\hat{+}}, \quad [\mathfrak{g}_0, \Omega^{\hat{+}\hat{+}}] = 2\Omega^{\hat{+}\hat{+}}, \quad (4.19)$$

then

$$(\text{ad}_{\mathfrak{g}_-})^N \left(\Omega^{\hat{+}\hat{+}} \eta_{\nu}(\tau_a^{m\hat{+}}, v_{a'}^{n\hat{+}}) \right) = 0 \quad \text{if} \quad N > 2 + \nu. \quad (4.20)$$

Hence for given J_{μ} and η_{ν} the condition $\langle 0 | \Lambda_N(\eta_{\nu}) J_{\mu}(Y|x) | 0 \rangle \neq 0$ demands

$$-2 - \nu \leq \mu \leq 2 + \nu, \quad 2 + \nu + \mu = 2N, \quad (4.21)$$

determining N unambiguously in terms of ν and μ .

Preforms $\Lambda_{gen} \in \mathbf{P}_{AdS}$ can be represented as

$$\Lambda_{gen} = \sum_N (\text{ad}_{\mathfrak{g}_-})^N \left(\Omega^{\hat{+}\hat{+}} \eta(\tau_a^{m\hat{+}}, v_{a'}^{n\hat{+}}) \right) c_N(\mathfrak{g}_0) \quad (4.22)$$

with some \mathfrak{g}_0 -dependent coefficients $c_N(\mathfrak{g}_0)$ encoding the freedom in normalization of charges.

The simplest choices $c_N = \frac{(\pm 1)^N}{N!} \alpha^\pm(g_0)$ with some $\alpha^\pm(g_0)$ yield the generating functions

$$\Lambda_{gen}^\pm = \exp \pm (\text{ad}_{g_-}) \left(\Omega^{\hat{+}\hat{+}} \eta^+(\tau_a^{m\hat{+}}, v_{a'}^{n\hat{+}}) \right) \alpha^\pm(g_0) \quad (4.23)$$

appropriate for the description of currents carrying positive and negative helicities.

Indeed, as mentioned in [17], any current $J(Y|x)$ can also be decomposed into a sum of currents $J^h(Y|x)$ of different *current helicities* h , that satisfy

$$[\mathcal{H}, J^h(Y|x)] = h J^h(Y|x), \quad (4.24)$$

where the *current helicity operator* is expressed in terms of g_j (3.32) as

$$\mathcal{H} = \frac{1}{4} (g_+ + g_-). \quad (4.25)$$

Using Eqs. (3.32), (3.33), (3.34), (4.1) and the Taylor expansion $f(x+y) = \exp(x \frac{\partial}{\partial y}) f(y)$, Eq. (4.23) yields

$$\Lambda_{gen}^\pm = \tilde{\Omega}^\pm \eta^\pm(\tilde{\tau}_a^m \pm, v_{a'}^n | g_0), \quad (4.26)$$

where $\eta^\pm(a, b | g_0) = \eta^\pm(a, b) \alpha^\pm(g_0)$,

$$\tilde{\tau}_a^m \pm = \tau_a^{m\hat{+}} \pm \tau_a^{m\hat{-}}, \quad \tilde{v}_{a'}^n \pm = v_{a'}^{n\hat{+}} \pm v_{a'}^{n\hat{-}}, \quad \tilde{\Omega}^\pm = \Omega^{\hat{+}\hat{+}} + \Omega^{\hat{-}\hat{-}} \pm (\Omega^{\hat{+}\hat{-}} + \Omega^{\hat{-}\hat{+}}). \quad (4.27)$$

The preforms Λ_{gen}^\pm (4.26) generate the space of nontrivial closed forms

$$\omega_+ = \langle \tilde{\Omega}^+ | \eta_+(\tilde{\tau}_a^m, \tilde{v}_{a'}^n | g_0) | J(Y|x) \rangle, \quad (4.28)$$

$$\omega_- = \langle \tilde{\Omega}^- | \eta_-(\tilde{\tau}_a^m, \tilde{v}_{a'}^n | g_0) | J(Y|x) \rangle \quad (4.29)$$

(and hence conserved currents) equivalent to the space of parameters depending on a single set of oscillators $\tilde{\tau}_a^m, \tilde{v}_{a'}^n$ in (4.28) and $\tilde{\tau}_a^m, \tilde{v}_{a'}^n$ in (4.29). As mentioned above, the dependence on g_0 in (4.28), (4.29) encodes the freedom in normalization of the original bilinear current (4.34).

Since

$$[\mathcal{H}, \tilde{\tau}_a^m \pm] = \pm \frac{1}{2} \tilde{\tau}_a^m \pm, \quad [\mathcal{H}, \tilde{v}_{a'}^n \pm] = \pm \frac{1}{2} \tilde{v}_{a'}^n \pm, \quad [\mathcal{H}, \tilde{\Omega}^\pm] = \pm \tilde{\Omega}^\pm, \quad (4.30)$$

ω_+ and ω_- (4.28) depend on the parameters carrying non-negative and non-positive current helicities, respectively.

Reformulating the result in the notations analogous (up to some rescalings) to those of [11] with

$$z^j_\alpha \rightarrow \partial^j_\alpha = \frac{\partial}{\partial y^\alpha_j}, \quad \bar{z}^j_{\alpha'} \rightarrow \partial^j_{\alpha'} = \frac{\partial}{\partial \bar{y}^{\alpha'}_j}, \quad \partial_{\pm\alpha} \sim \partial^1_{\alpha} \pm \partial^2_{\alpha}, \quad y_{\pm\alpha} \sim y_{1\alpha} \pm y_{2\alpha}, \quad \text{etc}, \quad (4.31)$$

$$\{\tilde{\tau}_a^m, \tilde{v}_{a'}^n\} \rightarrow \{\varrho, \bar{\varrho}\}, \quad \{\tilde{\tau}_a^m, \tilde{v}_{a'}^n\} \rightarrow \{\epsilon, \bar{\epsilon}\},$$

nontrivial charges are represented by the closed three-forms

$$\begin{aligned}\Omega_\eta(J) &= \mathcal{H}^{\alpha\alpha'} \partial_{-\alpha} \partial_{-\alpha'} \eta(\varrho, \bar{\varrho} | g_0) J(y^\pm, \bar{y}^\pm | x) \Big|_{y^\pm = \bar{y}^\pm = 0}, \\ \Omega_{\tilde{\eta}}(J) &= \mathcal{H}^{\alpha\alpha'} \partial_{+\alpha} \partial_{+\alpha'} \tilde{\eta}(\epsilon, \bar{\epsilon} | g_0) J(y^\pm, \bar{y}^\pm | x) \Big|_{y^\pm = \bar{y}^\pm = 0},\end{aligned}\tag{4.32}$$

where

$$\begin{aligned}\varrho_{-a} &= c_a^\alpha(x) \partial_{-\alpha} + s_a^{\alpha'}(x) \bar{y}^+_{\alpha'}, & \varrho^+_{+a} &= c_a^\alpha(x) y^+_\alpha + s_a^{\alpha'}(x) \bar{\partial}_{-\alpha'}, \\ \epsilon^-_{-a} &= c_a^\alpha(x) y^-_\alpha + s_a^{\alpha'}(x) \bar{\partial}_{+\alpha'}, & \epsilon^+_{+a} &= c_a^\alpha(x) \partial_{+\alpha} + s_a^{\alpha'}(x) \bar{y}^-_{\alpha'}\end{aligned}$$

and $c^\beta(x)$ and $s^{\beta'}(x)$ are Killing spinors (3.27), (3.28). $\bar{\varrho}$ and $\bar{\epsilon}$ are complex conjugated to ϱ and ϵ , respectively. In these variables the current helicity operator \mathcal{H} (4.25) is

$$\mathcal{H} = \frac{1}{2} \left(\bar{y}^+_{\alpha'} \bar{\partial}^+_{\alpha'} + y^+_\alpha \partial^+_\alpha - \bar{y}^-_{\alpha'} \bar{\partial}^-_{\alpha'} - y^-_\alpha \partial^-_\alpha \right).\tag{4.33}$$

For bilinear currents

$$J(y^\pm, \bar{y}^\pm | x) = C^+_{p_+}(y^+ + y^-, \bar{y}^+ + \bar{y}^- | x) C^-_{p_-}(y^+ - y^-, \bar{y}^+ - \bar{y}^- | x),\tag{4.34}$$

where the fields $C^\pm_{p_\pm}(y, \bar{y} | x)$ carry helicities p_\pm and solve rank-one equations (2.30)

$$D^{tw}_{ads} C^+_{p_+}(y, \bar{y} | x) = 0, \quad D^{tw}_{ads} C^-_{p_-}(iy, i\bar{y} | x) = 0,$$

Eqs. (4.32) yield the following two closed forms announced in [11]:

$$\begin{aligned}\mathcal{H}^{\alpha\alpha'} \frac{\partial}{\partial y^{-\alpha}} \frac{\partial}{\partial \bar{y}^{-\alpha'}} \eta(\varrho, \bar{\varrho} | p_+ - p_-) C^+_{p_+}(y^+ + y^-, \bar{y}^+ + \bar{y}^- | x) C^-_{p_-}(y^+ - y^-, \bar{y}^+ - \bar{y}^- | x) \Big|_{y^\pm = \bar{y}^\pm = 0}, \\ \mathcal{H}^{\alpha\alpha'} \frac{\partial}{\partial y^{+\alpha}} \frac{\partial}{\partial \bar{y}^{+\alpha'}} \tilde{\eta}(\epsilon, \bar{\epsilon} | p_+ - p_-) C^+_{p_+}(y^+ + y^-, \bar{y}^+ + \bar{y}^- | x) C^-_{p_-}(y^+ - y^-, \bar{y}^+ - \bar{y}^- | x) \Big|_{y^\pm = \bar{y}^\pm = 0},\end{aligned}\tag{4.35}$$

which represent two generating functions for gauge invariant conformal HS current cohomology in AdS_4 . By virtue of (4.30) and (4.31) the charges

$$\begin{aligned}Q^+_\eta &= \int \mathcal{H}^{\alpha\alpha'} \partial_{-\alpha} \partial_{-\alpha'} \eta(\varrho, \bar{\varrho} | p_+ - p_-) C^+_{p_+}(y^+ + y^-, \bar{y}^+ + \bar{y}^- | x) C^-_{p_-}(y^+ - y^-, \bar{y}^+ - \bar{y}^- | x) \Big|_{y^\pm = \bar{y}^\pm = 0}, \\ Q^-_{\tilde{\eta}} &= \int \mathcal{H}^{\alpha\alpha'} \partial_{+\alpha} \partial_{+\alpha'} \tilde{\eta}(\epsilon, \bar{\epsilon} | p_+ - p_-) C^+_{p_+}(y^+ + y^-, \bar{y}^+ + \bar{y}^- | x) C^-_{p_-}(y^+ - y^-, \bar{y}^+ - \bar{y}^- | x) \Big|_{y^\pm = \bar{y}^\pm = 0}\end{aligned}$$

are supported by the parameters of non-negative and non-positive current helicities, respectively. This list of charges matches the space of parameters of the $4d$ conformal HS symmetry algebra as discussed in [11].

5 Minkowski current cohomology

Analogous results for flat Minkowski space announced in [11] can be obtained as follows. In the limit $\lambda \rightarrow 0$, appropriately rescaled Minkowski current equations, resulting from (3.16) along with (3.18) and (3.17), take the form (see, e.g., [11] and references therein)

$$D_{2Mnk}^{tw} |J(Y|x)\rangle = 0, \quad D_{2Mnk}^{tw} = d + \widetilde{W}_{Mnk}^{(2)}, \quad (5.1)$$

$$\widetilde{W}_{Mnk}^{(2)} = e^{\alpha\beta'} \kappa_{\alpha}^{-\hat{n}} \zeta_{\beta' - \hat{n}}. \quad (5.2)$$

One can see that $f^{+-} \in {}^v\mathfrak{su}(2)$ (3.30) and the full algebra ${}^h\mathfrak{su}(2)$ (3.31) commute with D_{2Mnk}^{tw} . Minkowski Killing spinors in the Cartesian coordinate system with $D^L = d$

$$\tilde{c}_a^{\alpha} = \delta_a^{\alpha}, \quad \tilde{s}_{a'}^{\alpha'} = \delta_{a'}^{\alpha'}, \quad \tilde{c}_{a'}^{\alpha} = -x^{\alpha\alpha'} \varepsilon_{\alpha'a'}, \quad \tilde{s}_a^{\alpha'} = -x^{\alpha\alpha'} \varepsilon_{\alpha a} \quad (5.3)$$

obey

$$d\tilde{c}_a^{\alpha} = 0, \quad d\tilde{s}_a^{\beta'} + e^{\alpha\beta'} \tilde{c}_{a\alpha} = 0, \quad d\tilde{s}_{a'}^{\gamma'} = 0, \quad d\tilde{c}_{a'}^{\alpha} + e^{\alpha\beta'} \tilde{s}_{a\beta'} = 0. \quad (5.4)$$

Hence, one can choose the following basis of covariantly constant oscillators

$$\vartheta_{\gamma}^{+\hat{k}} = \tilde{c}_a^{\gamma}(x) \kappa_{\gamma}^{+\hat{k}} - \tilde{s}_a^{\gamma'}(x) \zeta_{\gamma'}^{+\hat{k}}, \quad \vartheta_{\gamma}^{-\hat{k}} = \tilde{c}_a^{\gamma}(x) \kappa_{\gamma}^{-\hat{k}}, \quad (5.5)$$

$$\varphi_{a'}^{-\hat{k}} = -\tilde{c}_{a'}^{\gamma}(x) \kappa_{\gamma}^{-\hat{k}} + \tilde{s}_{a'}^{\gamma'}(x) \zeta_{\gamma'}^{-\hat{k}}, \quad \varphi_{a'}^{+\hat{k}} = \tilde{s}_{a'}^{\gamma'}(x) \zeta_{\gamma'}^{+\hat{k}}. \quad (5.6)$$

Analogously to the AdS_4 case, in terms of (4.31), this gives that Minkowski nontrivial charges are fully represented by the following closed three-forms

$$\mathcal{H}^{\alpha\alpha'} \partial_{-\alpha} \partial_{-\alpha'} \eta(\xi, \bar{\xi}|g_0) J(y^{\pm}, \bar{y}^{\pm}|x) \Big|_{y^{\pm}=\bar{y}^{\pm}=0}, \quad (5.7)$$

$$\mathcal{H}^{\alpha\alpha'} \partial_{+\alpha} \partial_{+\alpha'} \eta(\chi, \bar{\chi}|g_0) J(y^{\pm}, \bar{y}^{\pm}|x) \Big|_{y^{\pm}=\bar{y}^{\pm}=0},$$

where J satisfies Minkowski current equations (5.1) and

$$\chi_{+\alpha} = \frac{\partial}{\partial y^{+\alpha}}, \quad \bar{\chi}_{+\beta'} = \frac{\partial}{\partial \bar{y}^{+\beta'}}, \quad \chi^{-\alpha} = y^{-\alpha} - x^{\alpha\beta'} \frac{\partial}{\partial \bar{y}^{+\beta'}}, \quad \bar{\chi}^{-\alpha'} = \bar{y}^{-\alpha'} - x^{\beta\alpha'} \frac{\partial}{\partial y^{+\beta}}, \quad (5.8)$$

$$\xi_{-\alpha} = \frac{\partial}{\partial y^{-\alpha}}, \quad \bar{\xi}_{-\beta'} = \frac{\partial}{\partial \bar{y}^{-\beta'}}, \quad \xi^{+\alpha} = y^{+\alpha} - x^{\alpha\beta'} \frac{\partial}{\partial \bar{y}^{-\beta'}}, \quad \bar{\xi}^{+\alpha'} = \bar{y}^{+\alpha'} - x^{\beta\alpha'} \frac{\partial}{\partial y^{-\beta}}. \quad (5.9)$$

As announced in [11], (5.7) represents two generating functions for the nontrivial current cohomology in Minkowski space, which are supported by the parameters of non-negative and non-positive current helicities, respectively.

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